

# Classes of Symmetric Cayley Graphs over Finite Abelian Groups of Degrees 4 and 6\*

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## Abstract

The present work is devoted to characterize the family of symmetric undirected Cayley graphs over finite Abelian groups for degrees 4 and 6.

## 1 Introduction

The Cayley graph over the group  $\Gamma$  and generating set  $S \subset \Gamma$ , denoted by  $\text{Cay}(\Gamma; S)$ , is defined as the graph with vertex set  $\Gamma$  and with set of adjacencies  $\{(x, xg) \mid x \in \Gamma, g \in S\}$ . If  $S = -S$ , then the graph is undirected. The present work is devoted to characterize the symmetric members of the family of undirected Cayley graphs over finite Abelian groups for degrees 4 and 6. Since these graphs are known to be vertex-transitive [2], the characterization will be done by determining those being edge-transitive.

In this paper the matricial notation by Fiol [4] for Cayley graphs over finite Abelian groups will be used. Hence, in order to be self-contained let us establish in this section the notation, definitions and results that will be used in this paper.

**Notation 1.** *The following notation will be used throughout the article:*

- Lower case letters denote integers:  $a, b, \dots$
- Bold font denotes integer column vectors:  $\mathbf{v}, \mathbf{w}, \dots$
- Capitals correspond to integral matrices:  $M, P, \dots$
- $\mathbf{e}_i$  denotes the vector with a 1 in its  $i$ -th component and 0 elsewhere.
- $\mathcal{B}_n = \{\mathbf{e}_i \mid i = 1, \dots, n\}$  denotes the  $n$ -dimensional orthonormal basis. Then,  $\pm\mathcal{B}_n = \{\pm\mathbf{e}_i \mid i = 1, \dots, n\}$ .

**Definition 2.** [4] Let  $M \in \mathbb{Z}^{n \times n}$  be a non-singular square matrix of dimension  $n$ . Two vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$  are congruent

modulo  $M$  if and only if we have  $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{Z}^n$  such that:  $\mathbf{v} - \mathbf{w} = u_1\mathbf{m}_1 + u_2\mathbf{m}_2 + \dots + u_n\mathbf{m}_n = M\mathbf{u}$ , where  $\mathbf{m}_j$  denotes the  $j$ -th column of  $M$ . We will denote this congruence as  $\mathbf{v} \equiv \mathbf{w} \pmod{M}$ .

Given a square non-singular integral matrix  $M \in \mathbb{Z}^{n \times n}$ , we will consider the Cayley graph  $\text{Cay}(\mathbb{Z}^n/M\mathbb{Z}^n; \pm\mathcal{B}_n)$ , hence:

- The vertex set is  $\mathbb{Z}^n/M\mathbb{Z}^n = \{\mathbf{v} \pmod{M} \mid \mathbf{v} \in \mathbb{Z}^n\}$ .
- Two vertices  $\mathbf{v}$  and  $\mathbf{w}$  are adjacent if and only if  $\mathbf{v} - \mathbf{w} \equiv \pm\mathbf{e}_i \pmod{M}$  for some  $i \in \{1, \dots, n\}$ .

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\*A previous version of some of the results in this paper were first announced at the 2010 International Workshop on Optimal Interconnection Networks (IWONT 2010)

From here onwards, all matrices will be considered to be non-singular, unless the contrary is stated. Note that, since  $\mathbb{Z}^n/M\mathbb{Z}^n$  has  $|\det(M)|$  elements, this will be the number of nodes of  $\text{Cay}(\mathbb{Z}^n/M\mathbb{Z}^n; \pm\mathcal{B}_n)$ . Moreover, since any vertex  $\mathbf{v}$  is adjacent to  $\mathbf{v} \pm \mathbf{e}_i \pmod{M}$ ,  $\text{Cay}(\mathbb{Z}^n/M\mathbb{Z}^n; \pm\mathcal{B}_n)$  is, in general, a regular graph of degree  $2n$ . Then, we call  $n$  the *dimension* of the graph. Note that when  $\mathbf{e}_i \equiv \pm\mathbf{e}_j \pmod{M}$  or  $2\mathbf{e}_i \equiv \mathbf{0} \pmod{M}$  for some  $1 \leq i, j \leq n$  then the degree of the Cayley graph is less than  $2n$ . In that case we can also consider the corresponding multigraph, which always has degree  $2n$ . The hypercube could be considered as an extreme case since  $\forall i \in \{1, \dots, n\}, 2\mathbf{e}_i \equiv \mathbf{0} \pmod{M}$  and therefore it has degree  $n$ .

The following result shows that considering  $\text{Cay}(\mathbb{Z}^n/M\mathbb{Z}^n; \pm\mathcal{B}_n)$  does not imply any loss of generality, that is, any Cayley graph over a finite Abelian group is isomorphic to  $\text{Cay}(\mathbb{Z}^n/M\mathbb{Z}^n; \pm\mathcal{B}_n)$ , for some matrix  $M \in \mathbb{Z}^{n \times n}$ .

**Theorem 3.** *For any connected Cayley graph  $G$  over a finite Abelian group there is  $M \in \mathbb{M}^{n \times n}$  non-singular such that  $G \cong \text{Cay}(\mathbb{Z}^n/M\mathbb{Z}^n; \pm\mathcal{B}_n)$ .*

*Proof.* Let the Cayley graph  $G = \text{Cay}(\Gamma; \{g_1, \dots, g_n\})$  with  $\Gamma$  Abelian and finite. We proceed by induction. If  $n = 1$  then  $M = (o(g_1))$ . Otherwise, let  $M_{n-1}$  be such that  $\text{Cay}(\mathbb{Z}/M_{n-1}\mathbb{Z}; \pm\mathcal{B}_{n-1}) \cong \text{Cay}(\Gamma; \{g_1, \dots, g_{n-1}\})$  with an isomorphism  $f(\mathbf{e}_i) = g_i$ . Then, let  $a$  be the minimum positive integer such that  $ag_n = x_1g_1 + x_2g_2 + \dots + x_{n-1}g_{n-1}$  for integers  $x_i$  (which exists because  $\Gamma$  is finite). Then  $M = \begin{pmatrix} M_{n-1} & \mathbf{x} \\ 0 & a \end{pmatrix}$  satisfies  $\text{Cay}(\mathbb{Z}^n/M\mathbb{Z}^n; \pm\mathcal{B}_n) \cong G$ .  $\square$

Hence, we will denote  $\text{Cay}(\mathbb{Z}^n/M\mathbb{Z}^n; \pm\mathcal{B}_n)$  by  $\mathcal{G}(M)$ . In [4] it was shown how performing different operations on matrix  $M$  remains the same graph. The next definitions and results recall this fact.

**Definition 4.** *Let  $M_1, M_2 \in \mathbb{Z}^{n \times n}$ . Then,  $M_1$  is right equivalent to  $M_2$ , denoted by  $M_1 \cong M_2$ , if and only if there exists a unit matrix  $P \in \mathbb{Z}^{n \times n}$  such that  $M_1 = M_2P$ .*

**Theorem 5.** [4]. *If matrices a pair of matrices  $M_1, M_2 \in \mathbb{Z}^{n \times n}$  are right-equivalent, then  $\mathcal{G}(M_1)$  and  $\mathcal{G}(M_2)$  are isomorphic graphs.*

**Definition 6.** *A signed permutation matrix is a matrix with entries in  $\{-1, 0, 1\}$  which has exactly one  $\pm 1$  in each row and column.*

Note that in  $\mathbb{Z}^{n \times n}$  the signed permutation matrices are exactly the unitary matrices, this is, the matrices  $U$  such  $UU^t = I$ . They are related to permutations in the way that for each permutation  $\sigma \in \Sigma_n$  there is a unique signed permutation matrix  $P_\sigma$  such that

$$P_\sigma \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \pm v_{\sigma(1)} \\ \vdots \\ \pm v_{\sigma(n)} \end{pmatrix}.$$

**Theorem 7.** [4] *If  $P$  is a signed permutation matrix then  $\mathcal{G}(PM) \cong \mathcal{G}(M)$ .*

In this paper we will find matrices such that  $\mathcal{G}(M)$  is edge-transitive for dimensions 2 and 3. In the first case, the characterization will be complete, that is we will find all  $\mathcal{G}(M)$  being symmetric. In the case of dimension 3, we will only consider those being edge-transitive by means of linear automorphism, as explained later.

With this aim, in Section 2 we will consider some properties of isomorphisms between Cayley graphs over finite Abelian groups, their automorphisms and the implications of containing cycles of length 4. In Section 3, we give some general results for the characterization of  $\mathcal{G}(M)$  graphs of any dimension symmetric by means of linear automorphisms. In Section 4 we give the full characterization of symmetric  $\mathcal{G}(M)$  graphs of dimension 2 (degree 4). In Section 5 we give the full characterization of symmetric by linear automorphisms  $\mathcal{G}(M)$  graphs of dimension 3 (degree 6).

## 2 Linear Automorphisms of Cayley Graphs $\mathcal{G}(M)$ and 4-cycles

Given a graph  $G = (V, E)$ ,  $\text{Aut}(G)$  denotes its automorphisms group.  $G$  is said to be *vertex-transitive* if, for any pair of vertices  $v_1, v_2 \in V$  there exists  $f \in \text{Aut}(G)$  such that  $f(v_1) = v_2$ . Similarly,  $G$  is said to be *edge-transitive* if for any pair of edges  $e_1 = (v_1, v_2), e_2 \in E$  there exists  $f \in \text{Aut}(G)$  such that  $f(e_1) = (f(v_1), f(v_2)) = e_2$ . Then, if  $G$  is both vertex and edge transitive, then it is called *symmetric*. The subgroup of  $\text{Aut}(G)$  of elements which fix some element  $x \in V$  is denoted as  $\text{Aut}(G, x)$  (also known as *stabilizer*).

All Cayley graphs are vertex-transitive [2]. The linear automorphisms of a Cayley graph  $\mathcal{G}(M)$  form a group  $L\text{Aut}(\mathcal{G}(M))$ . This group usually coincides with the full automorphism group  $\text{Aut}(\mathcal{G}(M))$ , except in a few cases that we consider separately. We also consider the group of linear automorphisms which fixes  $\mathbf{0}$ ,  $L\text{Aut}(\mathcal{G}(M), \mathbf{0})$ .

**Definition 8.**  $\mathcal{G}(M)$  is said linearly edge-transitive if for every  $i$  there exists  $f \in \text{LAut}(\mathcal{G}(M), \mathbf{0})$  such that  $f(\mathbf{e}_i) = \pm \mathbf{e}_i$ .

Clearly, a linearly edge-transitive Cayley graph  $\mathcal{G}(M)$  is symmetric. Therefore, in this section we study the automorphism group of  $\mathcal{G}(M)$  graphs. A basic question is determining when there are nonlinear automorphisms; which is very related to the problem of determining *Ádám isomorphism* [1, 3]. A pair of graphs  $\mathcal{G}(M_1)$  and  $\mathcal{G}(M_2)$  are *Ádám isomorphic* if there exists an isomorphism between their groups of vertices such that it sends the set of generators of one graph into the generators of the other graph. It is clear that any Ádám isomorphic graphs are isomorphic, but the opposite it is not always true.

In [3] it was proved that any pair of isomorphic Cayley multigraphs of degree four are Ádám isomorphic unless the pair is (up to Ádám isomorphism)  $\begin{pmatrix} 2k+1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2k & 2 \\ 0 & 2 \end{pmatrix}$  for some integer  $k$ . Using the nonlinear isomorphism between them one can build a nonlinear automorphism in each; hence they will appear in our study of the nonlinear automorphisms of dimension 2. However the reverse is not true, as there are a few graphs with nonlinear automorphisms which do not have a pairing non-Ádám isomorphic graph.

**Definition 9.** The neighborhood of a vertex  $v$  in the graph  $G = (V, E)$  is defined as  $N(v) = \{w \mid (v, w) \in E\}$ . Then, the common neighborhood of a list of vertices  $v_1, \dots, v_n \in V$  as  $N(v_1, \dots, v_n) = \bigcap_{i=1}^n N(v_i)$ .

**Theorem 10.** The neighborhood is preserved in graph isomorphisms. That is, if  $f$  is a graph isomorphism, then

$$N(f(v_1), \dots, f(v_n)) = \{f(w) \mid w \in N(v_1, \dots, v_n)\}.$$

*Proof.* Let  $f$  be a graph isomorphism from  $G = (V, E)$  into  $G' = (V', E')$ . We have that  $f(w) \in N(f(v_1), \dots, f(v_n))$  if only if  $\forall i, f(w) \in N(f(v_i))$ , that is  $\forall i, (f(w), f(v_i)) \in E'$ . Since  $f$  is an isomorphism we have that this is equivalent to  $\forall i, (w, v_i) \in E$  so  $w \in N(v_1, \dots, v_n)$ .  $\square$

Next, we analyze which isomorphisms between Cayley graphs over finite Abelian groups are linear mappings. This is related to the following concept.

**Definition 11.** We say that  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \pm \mathcal{B}_n$  form a 4-cycle in  $\mathcal{G}(M)$  if  $\mathbf{0} \equiv \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} \pmod{M}^\dagger$ . If we have  $\mathbf{a} \in \{-\mathbf{b}, -\mathbf{c}, -\mathbf{d}\}$  then we call the cycle trivial. Then, we say that  $\mathcal{G}(M)$  has not nontrivial 4-cycles if all its 4-cycles are trivial.

**Theorem 12.** If  $\mathcal{G}(M)$  has not nontrivial 4-cycles, then for all  $\mathbf{a}, \mathbf{b} \in \pm \mathcal{B}_n$  with  $\mathbf{a} \neq \mathbf{b}$

$$N(\mathbf{a}, \mathbf{b}) = \{\mathbf{0}, \mathbf{a} + \mathbf{b}\}.$$

*Proof.* If  $\mathbf{v} \in N(\mathbf{a}, \mathbf{b})$  then  $\exists \mathbf{x}, \mathbf{y} \in \pm \mathcal{B}_n$  such that  $\mathbf{v} = \mathbf{a} + \mathbf{x} = \mathbf{b} + \mathbf{y}$ . Since we have  $\mathbf{a} - \mathbf{b} + \mathbf{x} - \mathbf{y} = \mathbf{0}$  and  $\mathcal{G}(M)$  has not nontrivial 4-cycles, it must be fulfilled one of the following expressions:

- $\mathbf{a} = \mathbf{b}$  contradicting the hypothesis,
- $\mathbf{a} = -\mathbf{x}$  and thus  $\mathbf{v} = \mathbf{a} - \mathbf{a} = \mathbf{0}$ ,
- $\mathbf{a} = \mathbf{y}$  and thus  $\mathbf{v} = \mathbf{b} + \mathbf{y} = \mathbf{a} + \mathbf{b}$ .

$\square$

**Lemma 13.** If  $f$  is an automorphism of  $\mathcal{G}(M)$ , then for any  $\mathbf{t} \in \mathbb{Z}^n / M\mathbb{Z}^n$ ,  $f_{\mathbf{t}} : \mathbf{x} \mapsto f(\mathbf{t} + \mathbf{x}) - f(\mathbf{t})$  is an automorphism of  $\mathcal{G}(M)$  with  $f_{\mathbf{t}}(\mathbf{0}) = \mathbf{0}$ .

*Proof.* We have  $f_{\mathbf{t}}(\mathbf{0}) = f(\mathbf{t} + \mathbf{0}) - f(\mathbf{t}) = \mathbf{0}$ , thus  $f_{\mathbf{t}}$  fixes  $\mathbf{0}$ . Now if  $\mathbf{x} \in \mathbb{Z}^n / M\mathbb{Z}^n$  is adjacent to  $\mathbf{y} \in \mathbb{Z}^n / M\mathbb{Z}^n$  then  $\mathbf{t} + \mathbf{x}$  is adjacent to  $\mathbf{t} + \mathbf{y}$  and then as  $f$  is an automorphism we have that  $f(\mathbf{t} + \mathbf{x})$  is adjacent to  $f(\mathbf{t} + \mathbf{y})$ . Hence  $f_{\mathbf{t}}(\mathbf{x})$  is adjacent to  $f_{\mathbf{t}}(\mathbf{y})$ .  $\square$

**Lemma 14.** Let  $\mathcal{G}(M)$  be such that it has not nontrivial 4-cycles. Then for any  $f \in \text{Aut}(\mathcal{G}(M), \mathbf{0})$  we have that  $f(\mathbf{a} + \mathbf{b}) = f(\mathbf{a}) + f(\mathbf{b})$  for any  $\mathbf{a}, \mathbf{b} \in \pm \mathcal{B}_n$ .

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$^\dagger$ each of  $\{(\mathbf{v}, \mathbf{v} + \mathbf{a}, \mathbf{v} + \mathbf{a} + \mathbf{b}, \mathbf{v} + \mathbf{a} + \mathbf{b} + \mathbf{c}, \mathbf{v} + \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}) \mid \mathbf{v} \in \mathbb{Z}^n / M\mathbb{Z}^n\}$  is a cycle of length 4.

*Proof.* Let  $\mathbf{a}, \mathbf{b} \in \pm\mathcal{B}_n$ . First we prove the lemma for  $\mathbf{a} \neq \mathbf{b}$ . From Theorem 12 we get that  $N(\mathbf{a}, \mathbf{b}) = \{\mathbf{0}, \mathbf{a} + \mathbf{b}\}$ , hence by Theorem 10  $N(f(\mathbf{a}), f(\mathbf{b})) = \{f(\mathbf{0}), f(\mathbf{a} + \mathbf{b})\} = \{\mathbf{0}, f(\mathbf{a}) + f(\mathbf{b})\}$ . As  $f(\mathbf{0}) = \mathbf{0}$  we have that  $f(\mathbf{a} + \mathbf{b}) = f(\mathbf{a}) + f(\mathbf{b})$ .

Now note that since for any  $\mathbf{a} \in \pm\mathcal{B}_n$ ,  $\mathbf{a} \neq -\mathbf{a}$  we have that for any  $f \in \text{Aut}(\mathcal{G}(M), \mathbf{0})$ ,  $f(-\mathbf{a}) = -f(\mathbf{a})$ .

It remains to prove that  $f(2\mathbf{a}) = 2f(\mathbf{a})$ . Consider the automorphism  $f'$  defined by  $f'(\mathbf{v}) = f(\mathbf{a} + \mathbf{v}) - f(\mathbf{a})$  (it is an automorphism by Lemma 13). We have  $f'(-\mathbf{a}) = -f'(\mathbf{a})$ , hence  $f(\mathbf{a} - \mathbf{a}) - f(\mathbf{a}) = -(f(\mathbf{a} + \mathbf{a}) - f(\mathbf{a}))$ . Rearranging terms we obtain the desired  $f(2\mathbf{a}) = 2f(\mathbf{a})$ .  $\square$

**Lemma 15.** *If  $\forall \mathbf{a}, \mathbf{b} \in \pm\mathcal{B}_n$ ,  $f \in \text{Aut}(\mathcal{G}(M), \mathbf{0})$ ,  $f(\mathbf{a} + \mathbf{b}) = f(\mathbf{a}) + f(\mathbf{b})$  then every  $f \in \text{Aut}(\mathcal{G}(M), \mathbf{0})$  is a group automorphism of  $\mathbb{Z}^n/M\mathbb{Z}^n$ .*

*Proof.* First we prove that for all  $f \in \text{Aut}(\mathcal{G}(M), \mathbf{0})$  we have that

$$\forall \mathbf{t} \in \mathcal{G}(M), f(\mathbf{t} + \mathbf{a} + \mathbf{b}) = f(\mathbf{t} + \mathbf{a}) + f(\mathbf{t} + \mathbf{b}) - f(\mathbf{t}).$$

Let  $\mathbf{t} \in \mathcal{G}(M)$ . We define  $f_{\mathbf{t}}(\mathbf{v}) = f(\mathbf{t} + \mathbf{v}) - f(\mathbf{t})$ , by Lemma 13  $f_{\mathbf{t}} \in \text{Aut}(\mathcal{G}(M), \mathbf{0})$ . By hypothesis, we have  $\forall \mathbf{t} \in \mathcal{G}(M)$ ,  $f_{\mathbf{t}}(\mathbf{a} + \mathbf{b}) = f_{\mathbf{t}}(\mathbf{a}) + f_{\mathbf{t}}(\mathbf{b})$ , which implies  $\forall \mathbf{t} \in \mathcal{G}(M)$ ,  $f(\mathbf{t} + \mathbf{a} + \mathbf{b}) - f(\mathbf{t}) = f(\mathbf{t} + \mathbf{a}) - f(\mathbf{t}) + f(\mathbf{t} + \mathbf{b}) - f(\mathbf{t})$ .

We need to prove  $\forall n_i \in \mathbb{N}$ ,  $f(\sum_i n_i \mathbf{e}_i) = \sum_i n_i f(\mathbf{e}_i)$ . We proceed by induction in  $N = \sum_i n_i$ ; for  $N = 0, 1$  it is immediate. Now let  $\mathbf{v} = \sum_i n_i \mathbf{e}_i$  and  $\sum_i n_i = N + 1$ . Let  $u, v$  be any positive integers such that  $n_u + n_v \geq 2$ . Now, because of the first claim,  $f(\mathbf{v}) = f((\mathbf{v} - \mathbf{e}_u - \mathbf{e}_v) + \mathbf{e}_u + \mathbf{e}_v) = f((\mathbf{v} - \mathbf{e}_u - \mathbf{e}_v) + \mathbf{e}_u) + f((\mathbf{v} - \mathbf{e}_u - \mathbf{e}_v) + \mathbf{e}_v) - f(\mathbf{v} - \mathbf{e}_u - \mathbf{e}_v)$ . Applying the induction hypothesis we have that:  $f(\mathbf{v}) = (f(\mathbf{v} - \mathbf{e}_u - \mathbf{e}_v) + f(\mathbf{e}_u)) + (f(\mathbf{v} - \mathbf{e}_u - \mathbf{e}_v) + f(\mathbf{e}_v)) - f(\mathbf{v} - \mathbf{e}_u - \mathbf{e}_v) = f(\mathbf{v} - \mathbf{e}_u - \mathbf{e}_v) + f(\mathbf{e}_u) + f(\mathbf{e}_v)$ . Then as  $f(\mathbf{v} - \mathbf{e}_u - \mathbf{e}_v) = f(\sum_i n_i \mathbf{e}_i - \mathbf{e}_u - \mathbf{e}_v) = \sum_i n_i f(\mathbf{e}_i) - f(\mathbf{e}_u) - f(\mathbf{e}_v)$  we have that  $f(\mathbf{v}) = \sum_i n_i f(\mathbf{e}_i)$ .  $\square$

**Theorem 16.** *If the graph  $\mathcal{G}(M)$  has not nontrivial 4-cycles then any graph automorphism with  $f(\mathbf{0}) = \mathbf{0}$  is a group automorphism of  $\mathbb{Z}^n/M\mathbb{Z}^n$ .*

*Proof.* If there are not nontrivial 4-cycles then by Lemma 14 we have for any  $f \in \text{Aut}(\mathcal{G}(M), \mathbf{0})$  that  $f(\mathbf{a} + \mathbf{b}) = f(\mathbf{a}) + f(\mathbf{b})$  for any  $\mathbf{a}, \mathbf{b} \in \pm\mathcal{B}_n$ . Now we have linearity by Lemma 15.  $\square$

### 3 Edge-transitivity of Cayley graphs $\mathcal{G}(M)$ by Linear Automorphisms

In this section we will consider those graphs  $\mathcal{G}(M)$  such that any of its automorphisms is a linear mapping.

**Theorem 17.** *For any  $f \in \text{LAut}(\mathcal{G}(M), \mathbf{0})$  there exists a signed permutation matrix  $P$  such that  $\forall \mathbf{a} \in \mathbb{Z}^n/M\mathbb{Z}^n$ ,  $f(\mathbf{a}) = P\mathbf{a}$ .*

*Proof.* We define  $P$  as:

$$P_{i,j} = \begin{cases} 1 & \text{if } f(\mathbf{e}_j) = \mathbf{e}_i \\ -1 & \text{if } f(\mathbf{e}_j) = -\mathbf{e}_i \\ 0 & \text{otherwise} \end{cases}$$

having  $f(\mathbf{e}_i) = P\mathbf{e}_i$ . Let  $\mathbf{a} = \sum n_i \mathbf{e}_i$ .

$$f(\mathbf{a}) = \sum_i n_i f(\mathbf{e}_i) = \sum_i n_i P\mathbf{e}_i = P \sum_i n_i \mathbf{e}_i = P\mathbf{a}$$

$\square$

**Theorem 18.** *For any  $M \in \mathbb{Z}^{n \times n}$  the mapping  $f(\mathbf{x}) = P\mathbf{x}$  is a linear automorphism of  $\mathcal{G}(M)$  if and only if there exists  $Q \in \mathbb{Z}^{n \times n}$  such that  $PM = MQ$ .*

*Proof.* We prove first the left to right implication. As  $f$  must be well-defined, for all  $i$ ,  $\mathbf{0} = P\mathbf{0} \equiv PM\mathbf{e}_i \pmod{M}$ . And then exists  $\mathbf{q}_i$  such that  $PM\mathbf{e}_i = M\mathbf{q}_i$ , gathering all is together

$$PM = [PM\mathbf{e}_1, \dots, PM\mathbf{e}_n] = M[\mathbf{q}_1, \dots, \mathbf{q}_n] = MQ.$$

For the right to left implication; by Theorem 7  $f$  is an isomorphism from  $\mathcal{G}(M)$  into  $\mathcal{G}(PM) = \mathcal{G}(MQ)$ . Then by Theorem 5  $f$  is an automorphism of  $\mathcal{G}(M)$ .  $\square$

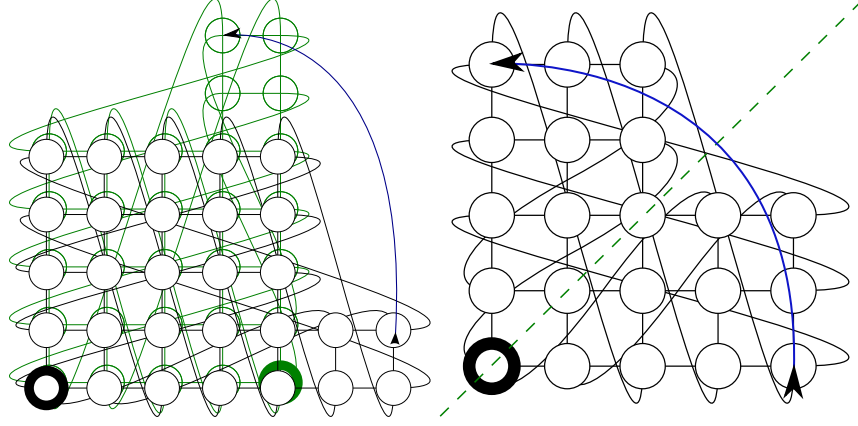


Figure 1: Linear Automorphisms of Cayley Graphs of Dimension 2.

To know if  $\mathcal{G}(M)$  is linearly edge-transitive we need to look to the multiplicative group of the signed permutation matrices  $P$  such  $PM = MQ$ . It is clear that, if a matrix representing a cycle of length  $n$  (even if it changes signs) is in the group then by composing it with itself, we can map  $\mathbf{e}_1$  to every  $\mathbf{e}_i$  making the graph edge-transitive. In these cases we have that  $LAut(\mathcal{G}(M), \mathbf{0})$  is a cyclic group. The smallest dimension for which we found  $LAut(\mathcal{G}(M), \mathbf{0})$  to be noncyclic is for  $n = 4$  with the Klein four-group. That situation occurs for example for Lipschitz graphs, which were introduced in in [5]. Since we just consider dimensions 2 and 3, this will not suppose any problem.

**Definition 19.** Two matrices  $A, B \in \mathbb{Z}^{n \times n}$  are similar, denoted by  $A \sim B$ , if there exists a unit matrix  $U \in \mathbb{Z}^{n \times n}$  such that  $AU = UB$ .

**Lemma 20.** Let  $PM = MQ$  and  $PM' = M'Q'$ . Then,  $M \cong M'$  if and only if  $Q \sim Q'$ .

*Proof.* Since  $PM = MQ$  and  $M = M'U$  then  $PM'U = M'UQ$  and  $PM' = M'(UQU^{-1}) = M'Q'$  with  $Q' \sim Q$ . Reciprocally, we know that if  $PM = MQ$  and  $Q' = UQU^{-1}$  then  $M' = MU$  produces  $PM' = M'Q'$  and  $M' \cong M$ .  $\square$

Since right equivalences leave the group invariant (Theorem 5), we know that for a given  $P$  we only need to see how many  $Q$  there are modulo similarity. Then, knowing  $P$  and  $Q$  we can solve for  $M$ . In [8] the next theorems are stated, which will be very helpful in the determination of  $Q$  in the following Sections 4 and 5.

**Theorem 21** ([8], Theorem III.12, page 50). Given a matrix  $A$  we can find a similar matrix, made of blocks, which is block upper triangular and moreover, that the blocks of the diagonal all have characteristic polynomial irreducible over  $\mathbb{Q}$ .

**Theorem 22** ([8], Theorem III.14, pag 53, The theorem of Lattimer and MacDuffee). If we have a matrix with irreducible characteristic polynomial, like the produced by the previous theorem then the number of matrices modulo similarity is the class number of  $\mathbb{Z}[\theta]$  where  $\theta$  is a root of the polynomial.

## 4 Characterization of Symmetric $\mathcal{G}(M)$ Graphs of Dimension 2

The complete characterization of symmetric  $\mathcal{G}(M)$  graphs with  $M \in \mathbb{Z}^{2 \times 2}$  will be done in this section. Firstly, we will consider those which are edge-transitive by means of linear automorphism. Later, we will consider those cases involving non-linear automorphisms.

By Theorem 17 a graph  $\mathcal{G}(M)$  is linearly edge-transitive if there is an automorphism  $f$  with  $f(\mathbf{e}_1) = \pm \mathbf{e}_2$  and  $f(\mathbf{e}_2) = \pm \mathbf{e}_1$ . Such automorphism is associated to one of the matrices:  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ .

Since  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  there is only need to check  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

**Theorem 23.** Let  $M \in \mathbb{Z}^{2 \times 2}$  be non-singular. Then,  $\mathcal{G}(M)$  is linearly edge-transitive if and only if, for some  $a, b \in \mathbb{Z}$ ,  $M$  is right equivalent to one of the following matrices:

$$M_1 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, M_2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, M_3 = \begin{pmatrix} a & -b \\ a & b \end{pmatrix}.$$

*Proof.* Let  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . We will determine  $Q$  and solve the system  $PM = MQ$ ; which by Theorem 18 is a necessary and sufficient condition to be linearly edge-transitive. The characteristic polynomial of  $\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  is  $\lambda^2 - 1$ , and the one of  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  is  $\lambda^2 + 1$ . As  $PM = MQ$  it must be the characteristic polynomial of both  $P$  and  $Q$ . By Lemma 20 we can choose any matrix similar to  $Q$  and obtain a matrix right-equivalent to  $M$ . Therefore, we have two cases:

- $P = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\lambda^2 - 1 = (\lambda + 1)(\lambda - 1)$ , being reducible over  $\mathbb{Q}$ , by Theorem 21  $Q$  must be similar to a matrix  $Q' = \begin{pmatrix} 1 & p \\ 0 & -1 \end{pmatrix}$ , which is similar to either  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  or to  $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . In the first case, depending on  $P$  we obtain  $M$  equal to  $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$  or to  $\begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \cong \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}$ ; which are the same under the variable change  $b \mapsto -b$ . In the second case, the same happens for the possible matrices  $\begin{pmatrix} a & -b \\ a & b \end{pmatrix}$  and  $\begin{pmatrix} a & b \\ -a & b \end{pmatrix} \cong \begin{pmatrix} b & -a \\ b & a \end{pmatrix}$  and the variable change  $a \mapsto b$ ,  $b \mapsto a$ .
- $P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ,  $\lambda^2 + 1$  which is irreducible over  $\mathbb{Q}$  and the class number of  $\mathbb{Z}[i]$  is 1, so by Theorem 22  $Q$  must be similar to  $P$ . The only possible solutions are  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ .

□

The first two cases of Theorem 23,  $\mathcal{G}(M_1)$  and  $\mathcal{G}(M_2)$ , are depicted in Figure 1. As it was proved in [6],  $\mathcal{G}(M_1)$  and  $\mathcal{G}(M_3)$  are isomorphic to the Kronecker product of two cycles. Furthermore  $\mathcal{G}(M_2)$  is isomorphic to the Gaussian graph introduced in [7].

#### 4.1 Edge-transitive $\mathcal{G}(M)$ Graphs of Dimension 2 by Nonlinear Automorphisms

In this subsection we focus on those  $\mathcal{G}(M)$  graphs with nontrivial 4-cycles, and hence, according to Theorem 16 their group of automorphisms could contain nonlinear automorphisms. Clearly, if there is a nontrivial 4-cycle then there exist  $\mathbf{a}, \mathbf{b} \in \pm \mathcal{B}_n$  which fulfill:

- i)  $4\mathbf{a} \equiv \mathbf{0} \pmod{M}$
- ii)  $3\mathbf{a} + \mathbf{b} \equiv \mathbf{0} \pmod{M}$
- iii)  $2\mathbf{a} + 2\mathbf{b} \equiv \mathbf{0} \pmod{M}$

If we consider  $u\mathbf{a} + v\mathbf{b} \equiv \mathbf{0} \pmod{M}$  it means that there exists  $\mathbf{x} \in \mathbb{Z}^2$  such that  $\mathbf{k} = \begin{pmatrix} u \\ v \end{pmatrix} = M\mathbf{x}$ . Now, let  $\gcd(\mathbf{x}) = \gcd(x_1, \dots, x_n)$ ,  $\mathbf{x}' = \frac{\mathbf{x}}{\gcd \mathbf{x}}$  and  $\mathbf{k}' = \frac{\mathbf{k}}{\gcd \mathbf{x}}$ , having  $\mathbf{k}' = M\mathbf{x}'$ . As  $\gcd \mathbf{x}' = 1$  we can build a unit matrix  $U$  with  $\mathbf{x}'$  as one of its columns, and therefore  $M' = MU$  has  $\mathbf{k}'$  as a column. In addition, Theorem 7 allows to choose each component positive.

We will begin with item (iii). In this case we obtain the matrix  $M = \begin{pmatrix} u & 2 \\ v & 2 \end{pmatrix}$ . If  $v = 2k$  we have that  $\begin{pmatrix} u & 2 \\ 2k & 2 \end{pmatrix}$  is right equivalent to  $\begin{pmatrix} u-v & 2 \\ 0 & 2 \end{pmatrix}$ . On the other hand, if  $v = 2k+1$  then  $\begin{pmatrix} u & 2 \\ 2k+1 & 2 \end{pmatrix}$  is right equivalent to  $\begin{pmatrix} u-v+1 & 2 \\ 1 & 2 \end{pmatrix}$ . Both matrices generate the same graph and there is a nonlinear isomorphism between them. In addition note that if the first column has odd weight then  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2k+1 & 2 \\ 0 & 2 \end{pmatrix} \cong \begin{pmatrix} 2k+2 & 2 \\ 1 & 2 \end{pmatrix}$ , so there is a linear isomorphism in addition to the nonlinear one. Hence, the only pairs non Ádám isomorphic are  $\left( \begin{pmatrix} 2k+1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2k & 2 \\ 0 & 2 \end{pmatrix} \right)$ , which correspond with the ones in [3]. Furthermore, the matrices of these non Ádám isomorphic graphs satisfy  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M \cong M$ , thus by Theorem 18 they are actually linearly edge-transitive.

For the items (i) and (ii) we begin proving that if there is exactly one nontrivial 4-cycle, then all automorphisms are linear. Furthermore note that these results are also valid in any number of dimensions.

**Lemma 24.** Let  $f \in \text{Aut}(\mathcal{G}(M), \mathbf{0})$  and  $\mathbf{a} \in \pm\mathcal{B}_n$ . If  $\forall \mathbf{b} \in \pm\mathcal{B}_n \setminus \{\mathbf{a}, -\mathbf{a}\}$ ,  $f(-\mathbf{b}) = -f(\mathbf{b})$  then  $f^{-1}(-\mathbf{a}) = -f^{-1}(\mathbf{a})$ .

*Proof.* We check three cases.

- Case  $f^{-1}(\mathbf{a}) \neq \pm\mathbf{a}$ . Applying the hypothesis we get  $\mathbf{a} = f(f^{-1}(\mathbf{a})) = -f(-f^{-1}(\mathbf{a}))$ . Then  $f^{-1}(-\mathbf{a}) = f^{-1}(f(-f^{-1}(\mathbf{a}))) = -f^{-1}(\mathbf{a})$ .
- Case  $f^{-1}(-\mathbf{a}) \neq \pm\mathbf{a}$ . Applying the hypothesis we get  $-\mathbf{a} = f(f^{-1}(-\mathbf{a})) = -f(-f^{-1}(-\mathbf{a}))$ . Then  $f^{-1}(\mathbf{a}) = f^{-1}(f(-f^{-1}(-\mathbf{a}))) = -f^{-1}(-\mathbf{a})$ .
- Case  $\{f^{-1}(\mathbf{a}), f^{-1}(-\mathbf{a})\} \subseteq \{\pm\mathbf{a}\}$ . As  $f^{-1}$  is a bijection we have the equality  $\{f^{-1}(\mathbf{a}), f^{-1}(-\mathbf{a})\} = \{\pm\mathbf{a}\}$ . Now  $f^{-1}(\mathbf{a}) + f^{-1}(-\mathbf{a}) = \mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ .

□

**Theorem 25.** If the only nontrivial 4-cycle is  $4\mathbf{a} \equiv \mathbf{0} \pmod{M}$  or  $3\mathbf{a} + \mathbf{b} \equiv \mathbf{0} \pmod{M}$  then

$$\text{Aut}(\mathcal{G}(M)) = \text{LAut}(\mathcal{G}(M)).$$

*Proof.* We proceed proving several claims iteratively.

- i) For all  $\mathbf{x}, \mathbf{y} \in \pm\mathcal{B}_n \setminus \{\mathbf{a}, -\mathbf{a}\}$ ,  $\mathbf{x} \neq \mathbf{y}$ ,  $N(\mathbf{x}, \mathbf{y}) = \{\mathbf{0}, \mathbf{x} + \mathbf{y}\}$ .

We have  $N(\mathbf{x}, \mathbf{y}) = \{\mathbf{v} \mid \mathbf{v} = \mathbf{x} + \mathbf{p} = \mathbf{y} + \mathbf{q}, \mathbf{p}, \mathbf{q} \in \pm\mathcal{B}_n\}$ . That is, we look for 4-cycles  $\mathbf{x} + \mathbf{p} - \mathbf{y} - \mathbf{q} = \mathbf{0}$ . The trivial ones are  $\mathbf{x} = -\mathbf{p}$  and  $\mathbf{x} = \mathbf{q}$  which respectively give  $\mathbf{v} = \mathbf{0}$  and  $\mathbf{v} = \mathbf{x} + \mathbf{y}$ . If it is the nontrivial 4-cycle  $4\mathbf{a} = \mathbf{0}$  then we have  $\{\mathbf{x}, \mathbf{y}\} = \{\mathbf{a}, -\mathbf{a}\}$ , contradicting the hypothesis. If it is the nontrivial 4-cycle  $3\mathbf{a} + \mathbf{b} = \mathbf{0}$ , then at least one of  $\mathbf{x}$  or  $\mathbf{y}$  is  $\pm\mathbf{a}$ .

- ii) For all  $\mathbf{x} \in \pm\mathcal{B}_n \setminus \{\mathbf{a}, -\mathbf{a}\}$ ,  $N(\mathbf{a}, \mathbf{x}) \subseteq \{\mathbf{0}, \mathbf{a} + \mathbf{x}, 2\mathbf{a}\}$ .

In this case we look for nontrivial 4-cycles  $\mathbf{a} + \mathbf{p} - \mathbf{x} - \mathbf{q} = \mathbf{0}$ . As  $\mathbf{x} \notin \{\pm\mathbf{a}\}$ , we have  $\mathbf{p} = -\mathbf{q} = \mathbf{a}$  and then  $\mathbf{v} = \mathbf{a} + \mathbf{p} = 2\mathbf{a}$ . Note that if we have the cycle  $4\mathbf{a} = \mathbf{0}$  then we only have the trivial solutions.

- iii)  $N(\mathbf{a}, -\mathbf{a}) = \{\mathbf{0}, \pm 2\mathbf{a}\}$ .

In this case we look for nontrivial 4-cycles  $\mathbf{a} + \mathbf{p} + \mathbf{a} - \mathbf{q} = \mathbf{0}$ . At least one of  $\mathbf{p}, -\mathbf{q}$  is equal to  $\mathbf{a}$ . If  $\mathbf{p} = \mathbf{a}$  then  $\mathbf{v} = \mathbf{a} + \mathbf{p} = 2\mathbf{a}$ . If  $-\mathbf{q} = \mathbf{a}$  then  $\mathbf{v} = -\mathbf{a} + \mathbf{q} = -2\mathbf{a}$ .

- iv) For all  $\mathbf{x} \in \pm\mathcal{B}_n \setminus \{\pm\mathbf{a}\}$ ,  $f \in \text{Aut}(\mathcal{G}(M), \mathbf{0})$ ,  $f(-\mathbf{x}) = -f(\mathbf{x})$ .

We have  $4\mathbf{x} \neq \mathbf{0}$ , since it would be another nontrivial 4-cycle. Hence  $\mathbf{x} \neq -\mathbf{x}$  and by item (i)  $N(\mathbf{x}, -\mathbf{x}) = \{\mathbf{0}\}$ . By Theorem 10 we have  $N(f(\mathbf{x}), f(-\mathbf{x})) = \{\mathbf{0}\}$ , thus  $f(\mathbf{x}) + f(-\mathbf{x}) = \mathbf{0}$ .

- v) For all  $\mathbf{x} \in \pm\mathcal{B}_n$ ,  $f \in \text{Aut}(\mathcal{G}(M), \mathbf{0})$ ,  $f(-\mathbf{x}) = -f(\mathbf{x})$  and  $f(2\mathbf{x}) = 2f(\mathbf{x})$ .

First apply Lemma 24 together item (iv) to  $f^{-1}$  to get  $\forall f \in \text{Aut}(\mathcal{G}(M), \mathbf{0})$ ,  $f(-\mathbf{a}) = -f(\mathbf{a})$ . Then considering the automorphism  $f'$  defined by  $f'(\mathbf{v}) = f(\mathbf{x} + \mathbf{v}) - f(\mathbf{x})$  like in the proof of Lemma 14 we obtain that  $f(2\mathbf{x}) = 2f(\mathbf{x})$ .

- vi) For all  $f \in \text{Aut}(\mathcal{G}(M), \mathbf{0})$ ,  $f(\pm\mathbf{a}) = \pm\mathbf{a}$ .

From item (iii) we have  $N(\mathbf{a}, -\mathbf{a}) = \{\mathbf{0}, \pm 2\mathbf{a}\}$  with  $\mathbf{0} \neq \pm 2\mathbf{a}$ . By Theorem 10 we get  $N(f(\mathbf{a}), f(-\mathbf{a})) = \{\mathbf{0}, f(\pm 2\mathbf{a})\}$  with  $\mathbf{0} \neq f(\pm 2\mathbf{a})$ . By items (i, ii, iii) we get  $\{f(\mathbf{a}), f(-\mathbf{a})\} = \{\pm\mathbf{a}\}$ .

- vii) For all  $f \in \text{Aut}(\mathcal{G}(M), \mathbf{0})$ ,  $\mathbf{x}, \mathbf{y} \in \pm\mathcal{B}_n$ ,  $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ .

If  $\mathbf{x} = \mathbf{y}$  it is item (v). Otherwise if neither of  $\mathbf{x}, \mathbf{y}$  is in  $\{\pm\mathbf{a}\}$  we proceed like the first step of the proof of Lemma 14; from item (i) we get  $N(\mathbf{x}, \mathbf{y}) = \{\mathbf{0}, \mathbf{x} + \mathbf{y}\}$ , hence by Theorem 10  $N(f(\mathbf{x}), f(\mathbf{y})) = \{f(\mathbf{0}), f(\mathbf{x} + \mathbf{y})\} = \{\mathbf{0}, f(\mathbf{x}) + f(\mathbf{y})\}$ . As  $f(\mathbf{0}) = \mathbf{0}$  we have that  $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ . Now if some is in  $\{\pm\mathbf{a}\}$ , we assume without loss of generality that  $\mathbf{y} = \mathbf{a}$  and  $\mathbf{x} \notin \{\pm\mathbf{a}\}$ . From item (ii) we have  $N(\mathbf{a}, \mathbf{x}) \subseteq \{\mathbf{0}, \mathbf{a} + \mathbf{x}, 2\mathbf{a}\}$ . And by Theorem 10 that  $N(f(\mathbf{a}), f(\mathbf{x})) \subseteq \{\mathbf{0}, f(\mathbf{a} + \mathbf{x}), f(2\mathbf{a})\}$ . By item (vi) we have  $N(f(\mathbf{a}), f(\mathbf{x})) \subseteq \{\mathbf{0}, f(\mathbf{a}) + f(\mathbf{x}), 2f(\mathbf{a})\}$ . As  $f(2\mathbf{a}) = 2f(\mathbf{a})$  (item (v)) we have that  $f(\mathbf{a} + \mathbf{x}) = f(\mathbf{a}) + f(\mathbf{x})$ .

- viii)  $\text{Aut}(\mathcal{G}(M)) = \text{LAut}(\mathcal{G}(M))$ .

Apply Lemma 15 to item (vii).

□

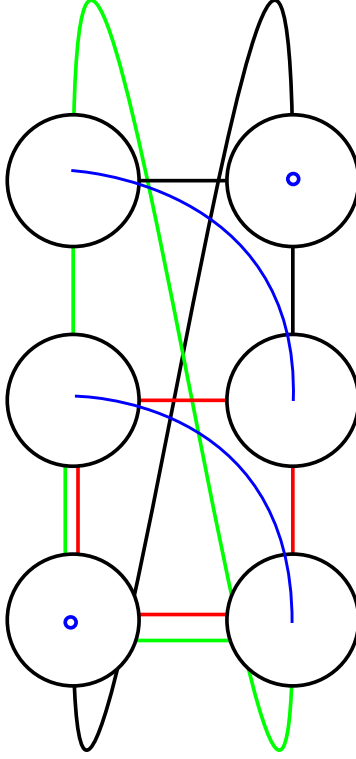


Figure 2: A nonlinear automorphism of  $\mathcal{G}(M)$ , where  $M = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ .

Finally, there are a few marginal cases in which the graph contains several nontrivial 4-cycles. These matrices are the matrices whose both columns correspond to nontrivial 4-cycles and their left divisors. These matrices can be built by selecting two columns in the set:

$$C = \left\{ \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}.$$

A complete study of the following cases, shows as that most of the combinations are edge-transitive. However, there are cases that lack of a nonlinear automorphism, leading to non-edge-transitive graphs.

Up to isomorphism, the bidimensional  $\mathcal{G}(M)$  graphs with 2 different nontrivial solutions for 4-cycles are:

- With nontrivial 4 cycles but without nonlinear automorphisms.

$$\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}$$

- With a nonlinear automorphism, which makes them edge-transitive,

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 1 & -1 \end{pmatrix} \cong \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$$

with an example in Figure 2. The first two have degree 3. Their associated Cayley multigraphs do not have nonlinear automorphisms. In the figure, we show in blue a nonlinear automorphism involution, which fixes two vertices and maps the nontrivial green 4-cycle into the red 4-cycle.

- With a nonlinear automorphism, but their linear automorphisms already make them edge-transitive,

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$$

with the torus as example in Figure 3.



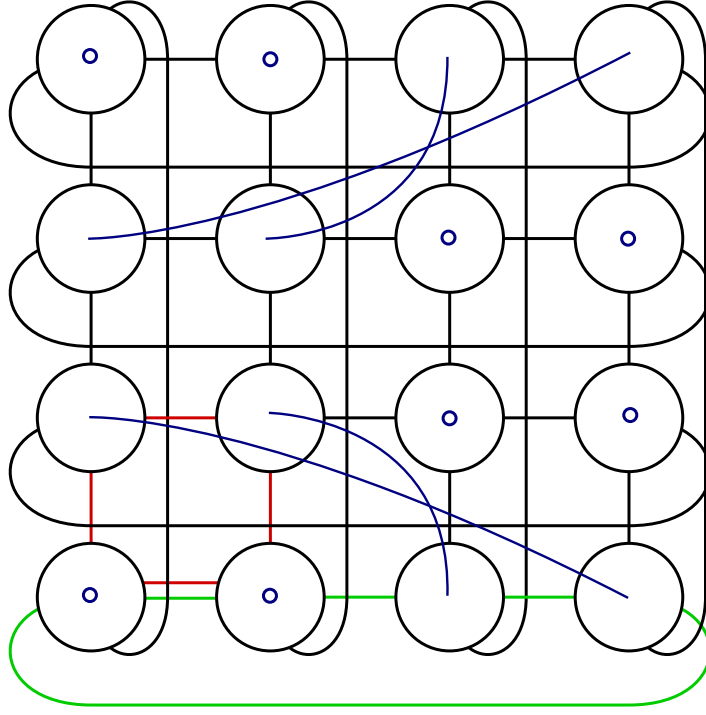


Figure 3: A nonlinear automorphism of the square torus of side 4.

## 5 Linearly Edge-Transitive $\mathcal{G}(M)$ Graphs of Dimension 3

This section provides a complete characterization of those  $\mathcal{G}(M)$  graphs with  $M \in \mathbb{Z}^{3 \times 3}$  being linearly edge-transitive.

**Lemma 26.** *Given  $M \in \mathbb{Z}^{3 \times 3}$ ,  $\mathcal{G}(M)$  is linearly edge-transitive if and only if there exists a signed permutation matrix of order 3 in  $LAut(\mathcal{G}(M), \mathbf{0})$ .*

*Proof.* If such a signed permutation exists, it is clear that  $\mathcal{G}(M)$  is linearly edge-transitive.

For the reciprocal, by Theorem 17 the automorphism is a signed permutation matrix. We can check that signed permutation matrices of dimension 3 can have orders 1, 2, 3, 4 and 6. The identity is the only signed permutation matrix of order 1 and it does not contribute to symmetry. Moreover, the signed permutation matrices which only change signs (that is, which are diagonal matrices) do not contribute to symmetry. Any remaining signed permutation matrix of orders 2 and 4 do not provide symmetry by themselves, since they fix one of the components, and the composition of two of them generates either a sign change or a signed permutation matrix of order 3 or 6.

Hence linear edge-transitivity implies the existence of an automorphism  $f \in LAut(\mathcal{G}(M), \mathbf{0})$  with order 3 or 6. If it has order 3, we already have the desired matrix. Otherwise we have  $f^3 = -id$  and so  $g = f^2$  has order 3.  $\square$

Hence, if  $\mathcal{G}(M)$  is linearly edge-transitive then  $LAut(\mathcal{G}(M), \mathbf{0})$  contains at least one of the next four cyclic groups as a subgroup and by Theorem 18 there is a matrix  $P$  such that  $PM = MQ$  for some  $Q$ .

$$\begin{aligned}
 P_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & P_2 &= \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \\
 P_3 &= \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} & P_4 &= \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
 \end{aligned}$$

These signed permutation matrices have characteristic and minimum polynomial  $\lambda^3 - 1$ . We can find some matrices (symbolic over 3 integer parameters) whose Cayley graphs are edge-transitive by taking  $Q = P$ , that is, we obtain  $M_i$  such that  $P_i M_i = M_i P_i$ . They are:

$$\begin{aligned}
M_1 &= \begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix}, & M_2 &= \begin{pmatrix} a & -c & -b \\ b & a & -c \\ c & b & a \end{pmatrix}, \\
M_3 &= \begin{pmatrix} a & -c & -b \\ b & a & c \\ c & -b & a \end{pmatrix}, & M_4 &= \begin{pmatrix} a & c & b \\ b & a & -c \\ c & -b & a \end{pmatrix}.
\end{aligned}$$

Next, we find the similar matrices.

**Lemma 27.** *There are exactly 2 similarity classes with characteristic polynomial  $\lambda^3 - 1$ :*

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \text{ and } Q_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

*Proof.* For  $\lambda^3 - 1 = (\lambda - 1)(\lambda(\lambda + 1) + 1)$  we have the following upper triangular block matrix which has it as its characteristic polynomial:  $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ . We know that

$$\begin{pmatrix} 1 & v & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u+2v & u-v \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -v & -u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

So  $\forall u, v \in \mathbb{Z}$ ,  $\begin{pmatrix} 1 & u+2v & u-v \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ . Since  $|\det(\begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix})| = 3$ , by Theorem 21, we have at most 3 matrices modulo similarity, which are:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

We check that the first two are non-similar. If

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

then

$$\begin{pmatrix} a & b & c \\ -d+g & -e+h & -f+i \\ -d & -e & -f \end{pmatrix} = \begin{pmatrix} a & -b-c & a+b \\ d & -e-f & d+e \\ g & -h-i & g+h \end{pmatrix}.$$

Hence  $d = g = 0$  and  $a = -3b$ ; and  $3b$  divides the determinant, which cannot be a unit. Now we see that the last two are similar.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

So we have proved that there are exactly 2 similarity classes with characteristic polynomial  $\lambda^3 - 1$ :

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \text{ and } Q_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

□

Finally, we explore the  $4 \cdot 2 = 8$  possible matrices from all the combinations.

**Lemma 28.** *With the previous definitions,  $P_1 \sim Q_2 \sim P_2 \sim P_3 \sim P_4$ .*

*Proof.* First we see that  $P_1 \sim Q_2$ .

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

And now that  $P_1 \sim P_2 \sim P_3 \sim P_4$ .

$$\begin{aligned} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

□

Thus, the first 4 matrices with  $P_i M = M Q_2$  are right equivalent to the previously calculated  $M_i$ . Therefore, we find the 4 symbolic matrices  $M'_i$  which satisfy  $P_i M'_i = M'_i Q_1$ .

$$\begin{aligned} M'_1 &= \begin{pmatrix} a & b & c \\ a & c & -b-c \\ a & -b-c & b \end{pmatrix} & M'_2 &= \begin{pmatrix} a & b & c \\ -a & -c & b+c \\ a & -b-c & b \end{pmatrix} \\ M'_3 &= \begin{pmatrix} a & b & c \\ a & c & -b-c \\ -a & b+c & -b \end{pmatrix} & M'_4 &= \begin{pmatrix} a & b & c \\ -a & -c & b+c \\ -a & b+c & -b \end{pmatrix} \end{aligned}$$

Now we have all the necessary elements to enunciate the tridimensional characterization of linearly edge-transitive graphs.

**Theorem 29.** *Let  $M \in \mathbb{Z}^{3 \times 3}$  be non-singular. Then,  $\mathcal{G}(M)$  is linearly edge-transitive if and only if it is isomorphic to  $\mathcal{G}(M_1)$  or  $\mathcal{G}(M'_1)$ , where:*

$$M_1 = \begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix} \text{ or } M'_1 = \begin{pmatrix} a & b & c \\ a & c & -b-c \\ a & -b-c & b \end{pmatrix}$$

for some  $a, b, c \in \mathbb{Z}$ .

*Proof.* Let  $\mathcal{G}(M)$  be linearly edge-transitive with  $M \in \mathbb{Z}^{3 \times 3}$ . By Lemma 26,  $P$  must exist with  $PM = MQ$  with  $P \in \{P_1, P_2, P_3, P_4\}$ . By Lemmas 20 and 27 there exist  $M'$  and  $Q$  with  $M \cong M'$ ,  $Q \in \{Q_1, Q_2\}$  and  $PM' = M'Q$ .

- If  $Q = Q_2$ , then by Lemma 28 we know  $M'' \in \{M_1, M_2, M_3, M_4\}$ , with  $PM'' = M''P$ ,  $M'' \cong M$ . Now we want to see that the matrices  $M_1, M_2, M_3$  and  $M_4$  generate the same set of matrices modulo graph-isomorphism. For each  $M_i$  we find a variable change and isomorphism from  $M_1$  into  $M_i$ :

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -a & c & b \\ b & -a & -c \\ c & -b & -a \end{pmatrix}$$

which is  $M_4$  giving  $a$  the value  $-a$ .

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M_1 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -a & c & -b \\ b & -a & c \\ -c & b & -a \end{pmatrix}$$

which is  $M_2$  giving  $a$  the value  $-a$  and  $c$  the value  $-c$ .

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} M_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} a & c & -b \\ b & a & -c \\ -c & -b & a \end{pmatrix}$$

which is  $M_3$  giving  $c$  the value  $-c$ .

- If  $Q = Q_1$ , then by Lemma 28 we know  $M' \in \{M'_1, M'_2, M'_3, M'_4\}$ . Now we want to see that the matrices  $M'_1, M'_2, M'_3$  and  $M'_4$  generate the same set of matrices modulo graph-isomorphism. For each  $M_i$  we find an isomorphism from  $M_1$  into  $M_i$ ; we do not need in this case variable changes:

$$M'_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M'_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} M'_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} M'_4$$

□

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